BONUS QUESTION: If $u = y' = \dfrac{dy}{dx}$, find an expression for $y^{(3)}$ in terms of only $y, u, \dfrac{du}{dy}$ and/or $\dfrac{d^2u}{dy^2}$. SCORE: _____ / 3 PTS

You may use the expression for y'' shown in lecture without proving it.

$$y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(u\frac{du}{dy}\right) = \underbrace{\frac{d}{dy}\left(u\frac{du}{dy}\right)\frac{dy}{dx}}_{\frac{1}{2}} = \left(\underbrace{\frac{du}{dy}\frac{du}{dy}}_{1} + \underbrace{u\frac{d^2u}{dy^2}}_{1}\right)\underbrace{u}_{\frac{1}{2}}$$

$$= u\left(\frac{du}{dy}\right)^2 + u^2\frac{d^2u}{dy^2}$$

Find the general solution of $e^y y'' = (y')^3$. SCORE: _____ / 8 PTS

$$u = y' \longrightarrow y'' = u\frac{du}{dy}$$

(2) $\underline{e^y u\frac{du}{dy} = u^3}$

(1) $\underline{u^{-2}du = e^{-y}dy}$

($1\frac{1}{2}$) $\underline{-u^{-1} = -e^{-y} + C}$

$$u = (e^{-y} + C)^{-1}$$

(1) $\underline{\frac{dy}{dx} = (e^{-y} + C)^{-1}}$

(1) $\underline{(e^{-y} + C)\,dy = dx}$

($1\frac{1}{2}$) $\underline{-e^{-y} + Cy + D = x}$

Use elimination **as shown in lecture** to solve the system of differential equations **SCORE: _____ / 22 PTS**

$$3x' + 4y' \quad - 6y = 3e^{2t} \qquad 3D[x] + (4D-6)[y] = 3e^{2t} \quad ①$$
$$2x' + 3y' - x - 5y = 4e^{2t} \qquad (2D-1)[x] + (3D-5)[y] = 4e^{2t} \quad ②$$

APPLY
2D-1 TO ①
-3D TO ②
AND ADD

$$((2D-1)(4D-6) - 3D(3D-5))[y] = 12e^{2t} - 3e^{2t} - 24e^{2t}$$
$$(-D^2 - D + 6)[y] = -15e^{2t}$$
$$② \ \underline{(D^2 + D - 6)[y]} = \underline{15e^{2t}} \ ②$$
$$r = 2, -3$$
$$① \ \underline{y_h = c_1 e^{2t} + c_2 e^{-3t}}$$
$$① \ \underline{y_p = Ate^{2t}}$$
$$y_p' = (2At + A)e^{2t}$$
$$y_p'' = (4At + 4A)e^{2t}$$
$$② \ \underline{y_p'' + y_p' - 6y_p = 5Ae^{2t}} = 15e^{2t}$$
$$A = 3$$
$$y = \underbrace{3te^{2t}}_{①} + \underbrace{c_1 e^{2t} + c_2 e^{-3t}}_{\frac{1}{2}}$$

APPLY
3D-5 TO ①
-(4D-6) TO ②
AND ADD

$$((3D-5)3D - (4D-6)(2D-1))[x] = 18e^{2t} - 15e^{2t} - 32e^{2t} + 24e^{2t}$$
$$\tfrac{1}{2} \ \underline{(D^2 + D - 6)[x]} = \underline{-5e^{2t}} \ ②$$
$$\tfrac{1}{2} \ \underline{x_h = k_1 e^{2t} + k_2 e^{-3t}}$$
$$\tfrac{1}{2} \ \underline{x_p = Bte^{2t}}$$
$$\tfrac{1}{2} \ \underline{x_p'' + x_p' - 6x_p = 5Be^{2t}} = -5e^{2t}$$
$$B = -1$$
$$x = \underbrace{-te^{2t}}_{①} + \underbrace{k_1 e^{2t} + k_2 e^{-3t}}_{\frac{1}{2}}$$

TALK TO ME IF YOU PLUGGED EVERYTHING BACK INTO THE 2ND ORIGINAL EQUATION

$$② \ 3(-2te^{2t} + (2k_1 - 1)e^{2t} - 3k_2 e^{-3t})$$
$$+4(6te^{2t} + (2c_1 + 3)e^{2t} - 3c_2 e^{-3t}) = \underline{(6k_1 + 2c_1 + 9)e^{2t}} \ ① = 3e^{2t}$$
$$-6(3te^{2t} + c_1 e^{2t} + c_2 e^{-3t}) \quad \underline{+(-9k_2 - 18c_2)e^{-3t}} \ ①$$

$$6k_1+2c_1+9=3 \longrightarrow k_1=-\tfrac{1}{3}c_1-1 \qquad \text{OR} \qquad c_1=-3k_1-3$$

$$-9k_2-18c_2=0 \longrightarrow k_2=-2c_2 \qquad\qquad c_2=-\tfrac{1}{2}k_2$$

$$x = -te^{2t} - \overbrace{\left(\tfrac{1}{3}c_1+1\right)e^{2t}}^{1} - \overbrace{2c_2e^{3t}}^{1}$$

$$y = 3te^{2t} + \underbrace{c_1e^{2t}+c_2e^{3t}}_{\frac{1}{2}}$$

$\frac{1}{2}$

OR

$$x = -te^{2t}+k_1e^{2t}+k_2e^{3t}$$

$$y = 3te^{2t}-(3k_1+3)e^{2t}-\tfrac{1}{2}k_2e^{3t}$$